*Determination of the Algebraic Curves whose Polar Conics are Parabolas.**

By Edward Kasner.

Among the curves of n^{th} order, one of the simplest classes, usually considered in connection with the theory of equations, is composed of those curves whose cartesian equation may be put into the form

$$y = a_0 x^n + a_1 x^{n-1} + \dots + a_n .$$

It is easily shown that the polar conic of any point in the plane with respect to such a curve is a parabola. The object of this note is to prove that this property is characteristic, i. e., that if any (non-decomposable) curve has the property stated, it may be reduced to the above form. The proof hinges upon the discussion of a partial differential equation, which is of interest also in connection with a class of developable surfaces.

Since the question considered belongs to metric geometry, it will be convenient to first derive the equations for polar curves in rectangular, instead of the usual homogeneous coordinates.

The first polar of the point x', y' with respect to the curve of n^{th} order $\phi(x, y) = 0$ is easily shown to be

$$x'\phi_x + y'\phi_y + n\phi - x\phi_x - y\phi_y = 0; \tag{1}$$

and the second polar is

$$\left.\begin{aligned} & x'^2\phi_{xx} + 2x'y'\phi_{xy} + y'^2\phi_{yy} \\ & + 2x'\{(n-1)\phi_x - (x\phi_{xx} + y\phi_{xy})\} + 2y'\{(n-1)\phi_y - (x\phi_{xy} + y\phi_{yy})\} \\ & + \{n(n-1)\phi - 2(n-1)(x\phi_x + y\phi_y) + x^2\phi_{xx} + 2xy\phi_{xy} + y^2\phi_{yy}\} = 0. \end{aligned}\right\} \tag{2}$$

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* Read before the American Mathematical Society, Dec. 28, 1901.

If x', y' are regarded as current coordinates, (6) represents the polar line of the point x, y and (7) represents the polar conic.

The polar conic will be a parabola provided

$$\Delta \equiv \phi_{xx}\phi_{yy} - \phi_{xy}^2 = 0. \tag{3}$$

Hence in connection with the general curve of n^{th} order we have a (metrically) related curve* $\Delta = 0$, of order $2(n-1)$, the locus of points whose polar conics are parabolas.

The problem before us may now be stated: *find the curves* $\phi = 0$ *for which* Δ *vanishes identically*, i. e., determine the rational integral solutions, of the n^{th} degree, of the partial differential equation *

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 0. \tag{4}$$

If ϕ is written in the form

$$\phi = u + R,$$

where u includes all the terms of n^{th} order and R the terms of lower order, the substitution in (4) gives as a necessary condition

$$u_{xx}u_{yy} - u_{xy}^2 = 0.$$

Since u is a binary form this indicates that it must be the n^{th} power of a linear form. Without loss of generality, we may write

$$\phi = x^n + R, \tag{5}$$

where R includes only terms whose degree is less than n.

The next step in the discussion is to show that if a function of form (5) satisfies (4), then R must be of the form

$$R = f(x) + cy, \tag{6}$$

where f is a polynomial in x of degree $n-1$ and c is a constant.

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* From the form of its equation, this might be termed the metric or cartesian Hessian of the original curve. Cf. C. A. Scott, "Note on the Real Inflexions of Plane Curves" (Transactions of the American Mathematical Society, Vol. 3, 1902, p. 398), where, more generally, the locus of points whose polar conics touch a given line is considered, and termed the *diacritic* of the line with respect to the curve.

† This is a Monge equation whose general solution may be obtained without difficulty; but this does not help in finding the rational integral solutions here required.

For the proof we employ the method of mathematical induction. We assume then that the result is correct when the order is $n-1$, and proceed to prove it for the order n.

In the first place, if the curve $\phi = 0$ belongs to the class considered, that is, if all its polar conics are parabolas, the same is evidently true of the polar curves

$$\xi\phi_x + \eta\phi_y + n\phi - x\phi_x - y\phi_y = 0.$$

These are curves of order $n-1$ to which the assumption may be applied. Take first the point $\xi = \infty$, $\eta = 0$, whose polar curve is

$$\phi_x \equiv nx^{n-1} + R_x = 0.$$

By the assumption made, the last term is of the form

$$R_x = f_{n-2}(x) + ky.$$

The integration of this gives

$$R = f(x) + kxy + g(y), \tag{7}$$

where f and g are of degree $n-1$.

Introducing this value in (5), and substituting in (4), the result should be an identity. Equating, in particular, the coefficient of ξ and the absolute term to zero, we obtain

$$\{n(n-1)(n-2)x^{n-3} + f'''\}(ng - yg')'' = 0, \tag{8}$$
$$(nf - xf')''(ng - yg')'' = (n-2)^2k^2, \tag{9}$$

where the accents indicate differentiation with respect to the variable involved.

Disregarding the cases $n = 1$ or 2, which may be considered by themselves without difficulty and taken as the starting point of the induction, it follows that the second factor of (8) must vanish. For since the degree of f is at most $n-1$, it is evident that the first factor cannot vanish. We have then

$$(ng - yg')'' = 0,$$

which, integrated twice, gives

$$ng - yg' = Ly + M.$$

We seek now for the integral of this equation whose degree does not exceed $n-1$. For this purpose substitute

$$g = b_0 y^{n-1} + b_1 y^{n-2} + \dots + b_{n-1},$$

obtaining the conditions

$$b_0 = b_1 = \dots = b_{n-3} = 0, \quad b_{n-2} = \frac{L}{n-1}, \quad b_{n-1} = \frac{M}{n-1}.$$

Hence g is of the form

$$g = ly + m.$$

Substituting this value of g in (9), it follows that $k = 0$. Hence (7) reduces to the form (6), and the latter is justified. This completes the induction.

From (5) and (6), the curve $\phi = 0$ may be written

$$x^n + f(x) + cy = 0. \tag{10}$$

If $c = 0$, this represents merely a set of parallel straight lines. Disregarding this trivial case, the equation may be written

$$y = F(x),$$

where F is a polynomial of n^{th} order.

If a curve of the n^{th} order has the property that all its polar conics are parabolas, then either it consists of n parallel straight lines, or it is of the form

$$y = a_0 x^n + a_1 x^{n-1} + \dots + a_n. \tag{11}$$

Conversely, both of these classes have the property in question, the polar conics in the first case being pairs of parallel lines, while in the second they are proper parabolas.

Incidentally, we have obtained, essentially, the rational integral solution of the differential equation (4). It is merely necessary to free the form (11) from the special choice of axes, introducing, for example, $ax + by$ and $by - ax$ instead of x and y respectively. *The general rational integral solution, of the n^{th} degree, of equation* (4) is

$$\phi = (hx + ky)^n + A_1 (hx + ky)^{n-1} + \dots + A_{n-2}(hx + ky)^2 + A_{n-1} x + A_n y + A_{n+1}, \tag{12}$$

thus involving $n + 3$ arbitrary constants.

If this is equated to zero, the result is the general curve having the property in question; it involves $n + 2$ parameters.*

* The degenerate case arises when the constants are connected by the relation $hA_n - kA_{n-1} = 0$.

The solution (12) may be applied to a question concerning developable surfaces. Consider the surfaces whose equation may be reduced to the form

$$z = F(x, y), \tag{13}$$

where F is a rational integral function of n^{th} order.

Such a surface will be developable when and only when, F is a solution of (9) *and hence of the form* (12). In this case the level curves cut out by the planes $z =$ constant will belong the class treated. Conversely if one (and hence all) of the level curves of the surface (13) belong to the class treated, then the surface is developable. The developable surfaces thus obtained are cylindrical.

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